Matrix Versions of some Classical Inequalities

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Dedicated to Maslina Darus, with friendship and respect

Abstract.

Some natural inequalities related to rearrangement in matrix products can also be regarded as extensions of classical inequalities for sequences or integrals. In particular, we show matrix versions of Chebyshev and Kantorovich type inequalities. The matrix approach may also provide simplified proofs and new results for classical inequalities. For instance, we show a link between Cassel's inequality and the basic rearrangement inequality for sequences of Hardy-Littlewood-Polya, and we state a reverse inequality to the Hardy-Littlewood-Polya inequality in which matrix technics are essential.

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Introduction

An important source of interesting inequalities in Matrix/operator theory is the study of rearrangements in a product. An obvious, but useful, example is the operator norm inequality

$$\|AB\|_\infty \le \|BA\|_\infty \tag{1}$$

whenever AB is normal. Here and in the sequel we use capital letters $A, B, \dots, Z$ to denote n-by-n complex matrices, or operators on a finite dimensional Hilbert space $\mathcal{H}$; I stands for the identity. When A is positive semidefinite, resp. positive definite, we write $A \ge 0$, resp. $A > 0$.

In a series of papers, the author showed further rearrangement inequalities companion to (1). These inequalities may be considered as matrix versions of some classical inequalities for sequences or integrals. The aim of this paper, which is mainly a survey, is to emphasize the link between these classical inequalities and matrix rearrangement inequalities. The concerned classical inequalities are of

Chebyshev and Kantorovich type. They can be stated for functions on a probability space, or equivalently for sequences. Let us present them for functions on an interval. In 1882, Chebyshev (see [11]) noted the following inequalities for bounded measurable functions f, g on a real interval Ω endowed with a probability measure μ: if f and g are both nondecreasing,

$$\int_\Omega f \,\mathrm{d}\mu \int_\Omega g \,\mathrm{d}\mu \leq \int_\Omega fg \,\mathrm{d}\mu. \tag{2}$$

Of course if f and $-g$ are both nondecreasing then the reverse inequality holds. For measurable functions f and g with $p \geq f(t) \geq q$ and $r \geq g(t) \geq s$, Gruss showed in 1934 (see [21]) the following estimate for the difference in Chebyshev inequality,

$$\left| \int_\Omega f \,\mathrm{d}\mu \int_\Omega g \,\mathrm{d}\mu - \int_\Omega fg \,\mathrm{d}\mu \right| \leq \frac{1}{4}(p-q)(r-s), \tag{3}$$

in particular if $a \geq f(t) \geq b$,

$$\int_\Omega f^2 \,\mathrm{d}\mu - \left(\int_\Omega f \,\mathrm{d}\mu \right)^2 \leq \frac{(a-b)^2}{4}. \tag{4}$$

Such an inequality is called a Kantorovich type inequality. Indeed, when $b \geq 0$, it is the additive version of

$$\int_\Omega f^2 \,\mathrm{d}\mu \leq \frac{(a+b)^2}{4ab} \left(\int_\Omega f \,\mathrm{d}\mu \right)^2 \tag{5}$$

which is equivalent to the original Kantorovich inequality stated in 1948 ([11]),

$$\int_\Omega f \,\mathrm{d}\mu \int_\Omega \frac{1}{f} \,\mathrm{d}\mu \leq \frac{(a+b)^2}{4ab}. \tag{6}$$

The fact that some matrix inequalities can be regarded as generalizations of integral inequalities (2)-(6) takes roots in the observation that these integral inequalities have immediate operator reformulations. By computing inner products in an orthonormal basis of eigenvectors for $Z > 0$ with extremal eigenvalues a and b, the Kantorovich inequality (6) may be rephrased as follows: for all norm one vectors h,

$$\langle h, Zh \rangle \langle h, Z^{-1}h \rangle \leq \frac{(a+b)^2}{4ab}.$$

Similarly taking square roots (5) can be rewritten

$$\|Zh\| \leq \frac{a+b}{2\sqrt{ab}} \, \langle h, Zh \rangle. \tag{7}$$

Such inner product inequalities are not only natural of their own right, they also motivate simple proofs via matrix techniques of the corresponding integral inequalities – or their discrete analogous for sequences. For instance an extremely simple proof of (7) [5] (see also [6]) is reproduced here as Lemma 2.2. A similar

remark holds for a nice paper of Yamaziki [24] about the Specht reverse arithmetic-geometric mean inequality.

Section 1 discusses an inequality for the Frobenius (or Hilbert-Schmidt) norm which is a matrix extension of Chebyshev's inequality and which implies several classical inequalities, in particular von Neumann's Trace inequality. In the next section we review several recent rearrangement inequalities for symmetric norms and eigenvalues which extend the Kantorovich inequalities. We also show a link between Cassel's inequality (a reverse Cauchy-Schwarz inequality) and a reverse inequality for the basic (Hardy-Littlewood-Polya) rearrangement inequality. Section 3 is concerned with matrix versions of generalized Kantorovich type inequalities.

1. Matrix Chebyshev inequalities

The results of this section originate from [3] (see also [6]). Here we give simpler proofs based on some matrices with nonnegative entries associated to normal operators. We also show the connection with standard inequalities.

Say that a pair of Hermitians (A, B) is monotone if there exists a third Hermitian C and two nondecreasing functions f and g such that $A = f(C)$ and $B = g(C)$. If g is nonincreasing (A, B) is antimonotone. We have the following result for the Frobenius (i.e., Hilbert-Schmidt) norm $\|\cdot\|_2$.

Theorem 1.1. *Let A, $B \geq 0$ and let Z be normal. If (A, B) is monotone,*

$$\|AZB\|_2 \leq \|ZAB\|_2.$$

If (A, B) is antimonotone,

$$\|AZB\|_2 \geq \|ZAB\|_2.$$

Note that, for positive A and B, (A, B) is monotone iff so is $(A^{1/2}, B^{1/2})$. Hence the first inequality is equivalent to

$$\mathrm{Tr}\, Z^*AZB \leq \mathrm{Tr}\, Z^*ZAB.$$

Since for Hermitians A, B, the pair (A, B) is monotone iff so is $(A + aI, B + bI)$ for any reals a, b, we then remark that Theorem 1.1 can be restated as trace inequalities involving Hermitian pairs:

Theorem 1.2. *Let A, B be Hermitian and let Z be normal. If (A, B) is monotone,*

$$\mathrm{Tr}\, Z^*AZB \leq \mathrm{Tr}\, Z^*ZAB.$$

If (A, B) is antimonotone,

$$\mathrm{Tr}\, Z^*AZB \geq \mathrm{Tr}\, Z^*ZAB.$$

These inequalities have found an application to quantum information theory [13].

Let us consider some classical facts related to these theorems. The case of Z unitary entails von-Neumann's Trace inequality,

$$|\mathrm{Tr}\ XY| \leq \sum \mu_j(X)\mu_j(Y) \tag{8}$$

where $\mu_j(\cdot)$, $j = 1, \ldots$ denotes the singular values arranged in decreasing order and counted with their multiplicities. This important inequality is the key for standard proofs of the Hölder inequality for Schatten p-norms. First if X and Y are both positive then, for some unitary Z, ZXZ^* and Y form a monotone pair and von Neumann's inequality follows from our theorems. For general X and Y, we consider polar decompositions $X = U|X|$, $Y = V|Y|$ and we note that

$$\begin{aligned} |\mathrm{Tr}\, XY| &= \left|\mathrm{Tr}\, |X^*|^{1/2}U|X|^{1/2}V|Y|\right| \\ &= \left|\mathrm{Tr}\, |Y|^{1/2}|X^*|^{1/2}U|X|^{1/2}V|Y|^{1/2}\right| \\ &\leq \{\mathrm{Tr}\, |Y||X^*|\}^{1/2}\{\mathrm{Tr}\, |X|V|Y|V^*\}^{1/2} \end{aligned}$$

by using the Cauchy-Schwarz inequality $|\mathrm{Tr}\, A^*B| \leq \{\mathrm{Tr}\, A^*A\}^{1/2}\{\mathrm{Tr}\, B^*B\}^{1/2}$ for all A, B. Hence the general case follows from the positive one.

Note that von Neumann's inequality is a matrix version of the classical Hardy-Littlewood rearrangement inequality: Given real scalars $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$,

$$\sum_{k=1}^n a_k^{\uparrow}b_k^{\downarrow} \leq \sum_{k=1}^n a_k b_k \leq \sum_{k=1}^n a_k^{\uparrow}b_k^{\uparrow} \tag{9}$$

where the exponent $\uparrow$ (resp. $\downarrow$) means the rearrangement in increasing (resp. decreasing) order. These inequalities also follow from Theorem 1.2 by letting Z be a permutation matrix and A, B be diagonal matrices.

Finally we remark that letting Z be a rank one projection, $Z = h \otimes h$, then we get in the monotone case

$$\|Ah\|\,\|Bh\| \leq \|ABh\| \quad \text{and} \quad \langle h, Ah\rangle\langle h, Bh\rangle \leq \langle h, ABh\rangle \tag{10}$$

for all unit vectors h. the reverse inequalities hold for antimonotone pairs. These are just restatements of Chebyshev's inequality (2).

The known proof of Theorems 1.1, 1.2 is quite intricated. The next lemma establishes a simple inequality for nonnegative matrices (i.e., with nonnegative entries) which entails the theorems. Our motivation for searching a proof via nonnegative matrices was the following observation. If $Z = (z_{i,j})$ is a normal matrix, then $X = (x_{i,j})$ with $x_{i,j} = |z_{i,j}|^2$ is a sum-symmetric matrix: for each index j,

$$\sum_k x_{k,j} = \sum_k x_{j,k}.$$

Indeed, the normality of Z entails $||Zh||^2 = ||Z^*h||^2$ for vectors h, in particular for vectors of the canonical basis.

For a nonnegative matrix X, we define its row-column ratio as the number

$$\mathrm{rc}(X) = \max_{1\le i\le n} \frac{\sum_k x_{i,k}}{\sum_k x_{k,i}},$$

whenever X has at least one nonzero entry on each column. If not, we set $\mathrm{rc}(X) = \lim_{r\to 0} \mathrm{rc}(X_r)$ where X_r is the same matrix as X, except that the zero entries are replaced by $r > 0$. It may happens that $\mathrm{rc}(X) = \infty$ and we adopt the convention $\infty \times 0 = \infty$.

Given real column vectors, $a = (a_1, \dots, a_n)^T$ and $b = (b_1, \dots, b_n)^T$ we denote by $a\cdot b$ the vector of the entrywise product of a and b. We denote the sum of the components of the vector a by $\sum a$. We say that a and b form a monotone (resp. antimonotone) pair if, for all indexes i, j,

$$a_i < a_j \Rightarrow b_i \le \text{(resp. } \ge) b_j,$$

equivalently

$$(a_i - a_j)(b_i - b_j) \ge \text{(resp. } \le) 0.$$

We may then state results which compare $a\cdot X(b)$ and $X(a\cdot b)$:

Lemma 1.3. *Let X be a nonnegative matrix and let (a, b) be a monotone pair of nonnegative vectors. Then, we have*

$$\sum a\cdot X(b) \le \mathrm{rc}(X) \sum X(a\cdot b)$$

and $\mathrm{rc}(X)$ is the best possible constant not depending on (a, b).

Lemma 1.4. *Let X be a real sum-symmetric matrix and let a and b be vectors. If (a, b) is monotone,*

$$\sum a\cdot X(b) \le \sum X(a\cdot b).$$

If (a, b) is antimonotone, the reverse inequality holds.

Proof of Lemma 1.3. Since $\mathrm{rc}(X) = \mathrm{rc}(PXP^T)$ for any permutation matrix P, and since

$$\sum a\cdot X(b) = \sum Pa\cdot PXP^T(Pb) \quad \text{and} \quad \sum X(a\cdot b) = \sum PXP^T(P(a)\cdot P(b)),$$

we may assume that both the components of a and b are arranged in increasing order. Let e_1 be the vector with all components 1, e_2 the vector with the first component 0 and all the others 1, ..., and e_n the vector with last component 1 and all the others 0. There exist nonnegative scalars α_j and β_j such that

$$a = \sum_{1\le j\le n} \alpha_j e_j \quad \text{and} \quad b = \sum_{1\le j\le n} \beta_j e_j.$$

By linearity, it suffices to show

$$\sum e_j\cdot X(e_k) \le \mathrm{rc}(X) \sum X(e_j\cdot e_k) \tag{11}$$

for all indexes j, k. We distinguish two cases.

If $j \le k$, then $X(e_j{\cdot}e_k) = X(e_k)$ and the inequality is obvious since $\sum e_j{\cdot}X(e_k) \le \sum X(e_k)$. If $j > k$, then using the definition of $\mathrm{rc}(X)$,

$$\begin{aligned}
\sum e_j\cdot X(e_k) &= \sum_{l=j}^{n}\sum_{m=k}^{n} x_{l,m}\\
&\le \sum_{l=j}^{n}\sum_{m=1}^{n} x_{l,m}\\
&\le \sum_{l=j}^{n}\mathrm{rc}(X)\sum_{m=1}^{n} x_{m,l}\\
&= \mathrm{rc}(X)\sum_{m=1}^{n}\sum_{l=j}^{n} x_{m,l}\\
&= \mathrm{rc}(X)\sum X(e_j)\\
&= \mathrm{rc}(X)\sum X(e_j\cdot e_k).
\end{aligned}$$

Hence (11) holds and the main part of Lemma 1.3 is proved. To see that this inequality is sharp, consider a vector u of the canonical basis correponding with an index i_0 such that

$$\mathrm{rc}(X) = \frac{\sum_k x_{i_0,k}}{\sum_k x_{k,i_0}}.$$

Then

$$\mathrm{rc}(X) = \frac{\sum X^T(u)}{\sum X(u)} = \frac{\sum uX(e_1)}{\sum X(ue_1)}$$

and (u, e_1) is monotone. This completes the proof. $\square$

Proof of Lemma 1.4. Since (a, b) is monotone iff $(a, -b)$ is antimonotone, it suffices to consider the first case. Fix a monotone pair (a, b) and a constant γ. Then (a, b) satisfies to the lemma iff the same holds for $(a+\gamma e_1, b+\gamma e_1)$. Hence we may suppose that (a, b) is nonnegative and we apply Lemma 1.3 with $\mathrm{rc}(X) = 1$. $\square$

Let us show how Theorem 1.1 (and similarly Theorem 1.2) follows from Lemma 1.4. Since (A, B) is monotone, we may assume that A and B are diagonal, $A = \mathrm{diag}(\alpha_1, \dots, \alpha_n)$ and $B = \mathrm{diag}(\beta_1, \dots, \beta_n)$. Let X be the sum-symmetric matrix $X = (x_{i,j})$ with $x_{i,j} = |z_{i,j}|^2$ and oberve that

$$\|AZB\|_2^2 = \sum a\cdot X(b) \quad \text{and} \quad \|ZAB\|_2^2 = \sum X(a\cdot b)$$

where $a = (\alpha_1^2 \dots \alpha_n^2)^T$ and $b = (\beta_1^2 \dots \beta_n^2)^T$ form a monotone or an antimonotone pair of vectors.

The following result [4] is another extension of (10). We omit the proof since it is contained in Theorem 2.1 below.

Theorem 1.5. *Let A, $B \geq 0$ with (A, B) monotone and let E be a projection. Then, there exists a unitary U such that*

$$|AEB| \leq U|ABE|U^*.$$

From this result we derived several eigenvalues inequalities and a determinantal Chebyshev type inequality involving compressions: *Let (A, B) be monotone positive and let $\mathcal{E}$ be a subspace. Then,*

$$\det A_{\mathcal{E}} \cdot \det B_{\mathcal{E}} \leq \det(AB)_{\mathcal{E}}.$$

Here $A_{\mathcal{E}}$ denotes the compression of A onto $\mathcal{E}$. When $\mathcal{E}$ has codimension 1, we showed the reverse inequality for antimonotone positive pairs (A, B),

$$\det A_{\mathcal{E}} \cdot \det B_{\mathcal{E}} \geq \det(AB)_{\mathcal{E}}.$$

Of course, this also holds for 1-dimensional subspaces as a restatement of (10). Hence we raised the following question:

Problem 1.6. Does the above determinantal inequality for antimonotone pairs hold for all subspaces?

We mention another open problem. We showed [3] that Theorem 1.2 can not be extended to Schatten p-norms when $p > 2$ by giving counterexamples in dimension 3. But the following is still open:

Problem 1.7. Does Theorem 1.1 hold for Schatten p-norms, $1 \leq p < 2$? In particular for the Trace norm?

1.1 Gruss type inequalities for the trace

In connection with Theorem 1.2 we have the following two results which are Gruss type inequalities for the trace. Letting Z be a rank one projection in the first result and assuming $AB = BA$ we get the classical Gruss inequalities (3), (4).

Proposition 1.8. *For $Z \geq 0$, Hermitian A with extremal eigenvalues p and q $(p \geq q)$ and Hermitian B with extremal eigenvalues r and s $(r \geq s)$,*

$$\left|\mathrm{Tr}\, Z^2AB - \mathrm{Tr}\, ZAZB\right| \leq \frac{1}{4}(p-q)(r-s)\mathrm{Tr}\, Z^2.$$

In particular,

$$\mathrm{Tr}\, Z^2A^2 - \mathrm{Tr}\,(ZA)^2 \leq \frac{(p-q)^2}{4}\mathrm{Tr}\, Z^2.$$

Proof. Note that
$$\mathrm{Tr}\, Z^2AB = \overline{\mathrm{Tr}\,(Z^2AB)^*} = \overline{\mathrm{Tr}\, Z^2BA}$$
and similarly
$$\mathrm{Tr}\, ZAZB = \overline{\mathrm{Tr}\,(ZBZA)}.$$
Since (A, A) is monotone, Theorem 1.2 shows that the map
$$(A, B) \longrightarrow \mathrm{Tr}\, Z^2AB - \mathrm{Tr}\, ZAZB$$
is a complex valued semi-inner product on the real vector space of Hermitian operators. The Cauchy-Schwarz inequality for this semi-inner product then shows that it suffices to prove the second inequality of our theorem. Let $Z = \sum_i z_i e_i \otimes e_i$ be the canonical expansion of Z. Since the Frobenius norm of a matrix is less than the l^2-norm of its diagonal, we have
$$\begin{aligned}\mathrm{Tr}\, Z^2A^2 - \mathrm{Tr}\,(ZA)^2 &= \mathrm{Tr}\, Z^2A^2 - \|Z^{1/2}AZ^{1/2}\|_2^2 \\ &\le \sum_i z_i^2\langle e_i, A^2e_i\rangle - \sum_i (z_i\langle e_i, Ae_i\rangle)^2 \\ &\le \frac{(p-q)^2}{4}\sum_i z_i^2 = \frac{(p-q)^2}{4}\mathrm{Tr}\, Z^2\end{aligned}$$
by using the classical inequality (4). □

Letting Z be a rank one projection we recapture an inequality pointed out by M. Fujii et al. [15],
$$|\langle h, ABh\rangle - \langle h, Ah\rangle\langle h, Bh\rangle| \le \frac{1}{4}(p-q)(r-s)$$
for all norm one vectors h. They called it the Variance-covariance Inequality. By using the GNS construction, this can be formulated in the C^*-algebra framework: Given positive elements a, b with spectra in $[p, q]$ and $[r, s]$ respectively,
$$|\varphi(ab) - \varphi(a)\varphi(b)| \le \frac{1}{4}(p-q)(r-s)$$
for all states φ.

Proposition 1.9. *For normal Z, Hermitian A with extremal eigenvalues p and q $(p \ge q)$ and Hermitian B with extremal eigenvalues r and s $(r \ge s)$,*
$$\left|\mathrm{Tr}\, |Z|^2AB - \mathrm{Tr}\, Z^*AZB\right| \le \frac{1}{2}(p-q)(r-s)\mathrm{Tr}\, |Z|^2.$$

Proof. Theorem 1.2 shows that the map
$$(A, B) \longrightarrow \mathrm{Tr}\, |Z|^2AB - \mathrm{Tr}\, Z^*AZB$$
is a semi-inner product on the space of Hermitian operators. Hence it suffices to consider the case $A = B$:

Let

$$\tilde{Z} = \begin{pmatrix} 0 & Z^* \\ Z & 0 \end{pmatrix} \qquad \text{and} \qquad \tilde{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

and observe that

$$\mathrm{Tr}\,|Z|^2A^2 - \mathrm{Tr}\,Z^*AZA = \frac{1}{2}\{\,\mathrm{Tr}\,\tilde{Z}^2\tilde{A}^2 - \mathrm{Tr}\,(\tilde{Z}\tilde{A})^2\,\}$$

and

$$\mathrm{Tr}\,|Z|^2 = \frac{1}{2}\mathrm{Tr}\,\tilde{Z}^2.$$

Consequently, $\tilde{Z}$ being Hermitian, we may assume so is Z. Replacing if necessary A by $A+qI$, we may also assume $A \geq 0$. We compute in respect with an orthonormal basis of eigenvectors for $A = \sum_i a_i e_i \otimes e_i$,

$$\begin{aligned}
\mathrm{Tr}\,Z^2A^2 - \mathrm{Tr}\,(ZA)^2 &= \mathrm{Tr}\,Z^2A^2 - \|A^{1/2}ZA^{1/2}\|_2^2 \\
&= \sum_{i,j} a_i^2|z_{i,j}|^2 - \sum_{i,j} a_ia_j|z_{i,j}|^2 \\
&= \sum_{i<j}(a_i - a_j)^2|z_{i,j}|^2 \\
&\leq \frac{(p-q)^2}{2}\mathrm{Tr}\,Z^2
\end{aligned}$$

and the proof is complete. $\square$

2. Matrix Kantorovich inequalities

In view of Theorem 1.5 involving projections, we tried to obtain an extension for all positive operators. We obtained a hybrid Chebyshev/Kantorovich result:

Theorem 2.1. *Let A, $B \geq 0$ with (A, B) monotone and let $Z \geq 0$ with its largest and smallest nonzero eigenvalues a and b. Then, there exists a unitary U such that*

$$|AZB| \leq \frac{a+b}{2\sqrt{ab}}U|ZAB|U^*.$$

Since for a projection Z, we have $a = b = 1$, Theorem 2.1 contains Theorem 1.5.

We recall that the inequality of Theorem 2.1 is equivalent to:

$$\mu_j(AZB) \leq \mu_j(ZAB)$$

for all $j = 1, \ldots,$ where $\{\mu_j(\cdot)\}$ stand for the singular values arranged in decreasing order with their multiplicities [2, p. 74].

We need some lemmas. First we state the Kantorovich inequality (7) again and we give a matrix proof.

Lemma 2.2. *Let $Z > 0$ with extremal eigenvalues a and b. Then, for every norm one vector h,*

$$\|Zh\| \le \frac{a+b}{2\sqrt{ab}}\,\langle h, Zh\rangle.$$

Proof. Let $\mathcal{E}$ be any subspace of $\mathcal{H}$ and let a' and b' be the extremal eigenvalues of $Z_{\mathcal{E}}$. Then $a \ge a' \ge b' \ge b$ and, setting $t = \sqrt{a/b}$, $t' = \sqrt{a'/b'}$, we have $t \ge t' \ge 1$. Since $t \longrightarrow t + 1/t$ increases on $[1, \infty)$ and

$$\frac{a+b}{2\sqrt{ab}} = \frac{1}{2}\left(t + \frac{1}{t}\right), \qquad \frac{a'+b'}{2\sqrt{a'b'}} = \frac{1}{2}\left(t' + \frac{1}{t'}\right),$$

we infer

$$\frac{a+b}{2\sqrt{ab}} \ge \frac{a'+b'}{2\sqrt{a'b'}}.$$

Therefore, it suffices to prove the lemma for $Z_{\mathcal{E}}$ with $\mathcal{E} = \mathrm{span}\{h, Zh\}$. Hence, we may assume $\dim \mathcal{H} = 2$, $Z = ae_1 \otimes e_1 + be_2 \otimes e_2$ and $h = xe_1 + (\sqrt{1-x^2})e_2$. Setting $x^2 = y$ we have

$$\frac{||Zh||}{\langle h, Zh\rangle} = \frac{\sqrt{a^2y + b^2(1-y)}}{ay + b(1-y)}.$$

The right hand side attains its maximum on $[0, 1]$ at $y = b/(a + b)$, and then

$$\frac{||Zh||}{\langle h, Zh\rangle} = \frac{a+b}{2\sqrt{ab}}$$

proving the lemma. $\square$

Lemma 2.2 can be extended as an inequality involving the operator norm $\|\cdot\|_\infty$ and the spectral radius $\rho(\cdot)$. Indeed, letting $A = h \otimes h$ in the next lemma, we get Lemma 2.2.

Lemma 2.3. *For $A \ge 0$ and $Z > 0$ with extremal eigenvalues a and b,*

$$\|AZ\|_\infty \le \frac{a+b}{2\sqrt{ab}}\,\rho(AZ).$$

Proof. There exists a rank one projection F such that, letting f be a unit vector in the range of $A^{1/2}F$,

$$\begin{aligned}\|AZ\|_\infty = \|ZA\|_\infty = \|ZAF\|_\infty &= \|ZA^{1/2}(f \otimes f)A^{1/2}F\|_\infty \\ &\le \|ZA^{1/2}(f \otimes f)A^{1/2}\|_\infty = \|A^{1/2}f\|^2 \|Z\frac{A^{1/2}f}{\|A^{1/2}f\|}\|.\end{aligned}$$

Hence

$$\|AZ\|_\infty \le \frac{a+b}{2\sqrt{ab}}\,\langle f, A^{1/2}ZA^{1/2}f\rangle \le \frac{a+b}{2\sqrt{ab}}\,\rho(A^{1/2}ZA^{1/2}) = \frac{a+b}{2\sqrt{ab}}\,\rho(AZ)$$

by using Lemma 2.2 with $h = A^{1/2}f/\|A^{1/2}f\|$. $\square$

From Lemma 2.3 one may derive a sharp operator inequality:

Lemma 2.4. *Let $0 \le A \le I$ and let $Z > 0$ with extremal eigenvalues a and b. Then,*

$$AZA \le \frac{(a+b)^2}{4ab}\, Z.$$

Proof. The claim is equivalent to the operator norm inequalities

$$\|Z^{-1/2}AZAZ^{-1/2}\|_\infty \le \frac{(a+b)^2}{4ab}$$

or

$$\|Z^{-1/2}AZ^{1/2}\|_\infty \le \frac{a+b}{2\sqrt{ab}}.$$

But the previous lemma entails

$$\begin{aligned}\|Z^{-1/2}AZ^{1/2}\|_\infty &= \|Z^{-1/2}AZ^{-1/2}Z\|_\infty \\ &\le \frac{a+b}{2\sqrt{ab}}\,\rho(Z^{-1/2}AZ^{-1/2}Z) \\ &= \frac{a+b}{2\sqrt{ab}}\,\|A\|_\infty \\ &\le \frac{a+b}{2\sqrt{ab}},\end{aligned}$$

hence, the result holds. $\square$

Proof of Theorem 2.1. We will use Lemma 2.4 and the following operator norm inequality

$$\|AEB\|_\infty \le \|ABE\|_\infty \tag{12}$$

for all projections E. This inequality was derived [3] from Theorem 1.1 and is the starting point and a special case of Theorem 1.5. In fact (12) is a consequence of (10). Indeed there exist unit vectors f and h with $h = Eh$ such that

$$\|AEB\|_\infty = \|AEBf\| = \|A(h\otimes h)Bf\| \le \|A(h\otimes h)B\|_\infty = \|Ah\|\|Bh\|$$

so that using (10)

$$\|AEB\|_\infty \le \|ABh\| \le \|ABE\|_\infty.$$

We denote by $\mathrm{supp}(X)$ the support projection of an operator X, i.e., the smallest projection S such that $X = XS$.

By the minimax principle, for every projection F, $\mathrm{corank}F = k-1$,

$$\begin{aligned}\mu_k(AZB) &\le \|AZBF\|_\infty \\ &= \|AZ^{1/2}EZ^{1/2}BF\|_\infty \\ &\le \|AZ^{1/2}EZ^{1/2}B\|_\infty\end{aligned} \tag{13}$$

where E is the projection onto the range of $Z^{1/2}BF$. Note that there exists a rank one projection P, $P \leq E$, such that

$$\mu_k(AZB) \leq \|AZ^{1/2}PZ^{1/2}B\|_\infty.$$

Indeed, let h be a norm one vector such that

$$\|AZ^{1/2}EZ^{1/2}B\|_\infty = \|AZ^{1/2}EZ^{1/2}Bh\|$$

and let P be the projection onto $\mathrm{span}\{EZ^{1/2}Bh\}$. Since $Z^{1/2}PZ^{1/2}$ has rank one, and hence is a scalar multiple of a projection, (12) entails

$$\mu_k(AZB) \leq \|Z^{1/2}PZ^{1/2}AB\|_\infty.$$

We may choose F in (13) in order to obtain any projection $G \geq \mathrm{supp}(EZ^{1/2}AB)$, $\mathrm{corank} G = k-1$. Since

$$\mathrm{supp}(PZ^{1/2}AB) \leq \mathrm{supp}(EZ^{1/2}AB) \leq G,$$

we infer

$$\mu_k(AZB) \leq \|Z^{1/2}PZ^{1/2}ABG\|_\infty.$$

Consequently, using Lemma 2.4 with Z and PZP,

$$\begin{aligned}\mu_k(AZB) &= \|GABZ^{1/2}PZPZ^{1/2}ABG\|_\infty^{1/2} \\ &\leq \frac{a+b}{2\sqrt{ab}}\,\|ZABG\|_\infty.\end{aligned}$$

Since we may choose G so that $\|ZABG\|_\infty = \mu_k(ZAB)$, the proof is complete. $\square$

Under an additional invertibility assumption on Z, Theorem 2.1 can be reversed:

Theorem 2.5. *Let A, $B \geq 0$ with (A,B) monotone and let $Z > 0$ with extremal eigenvalues a and b. Then, there exists a unitary V such that*

$$|ZAB| \leq \frac{a+b}{2\sqrt{ab}}\,V|AZB|V^*$$

Proof. By a limit argument we may assume that both A and B are invertible. Hence, taking inverses in Theorem 2.1 considered as singular values inequalities, we obtain a unitary W (actually we can take $W = U$ since $t \longrightarrow t^{-1}$ is operator decreasing) such that

$$|AZB|^{-1} \geq \frac{2\sqrt{ab}}{a+b}\,W|ZAB|^{-1}W^*$$

hence, using $|X|^{-1} = (X^*X)^{-1/2} = (X^{-1}X^{*-1})^{1/2} = |X^{*-1}|$ for all invertibles X,

$$|B^{-1}Z^{-1}A^{-1}| \geq \frac{2\sqrt{ab}}{a+b}\,W|A^{-1}B^{-1}Z^{-1}|W^*.$$

Then observe that we can replace Z^{-1} by Z since

$$\frac{a+b}{2\sqrt{ab}} = \frac{a^{-1}+b^{-1}}{2\sqrt{a^{-1}b^{-1}}}.$$

As the correspondence between an invertible monotone pair and its inverse is onto, Theorem 2.5 holds. $\square$

Let X with real eigenvalues and denote by $\lambda_k(X)$, $k = 1, 2, \dots$, the eigenvalues of X arranged in decreasing order with their multiplicities. Replacing A and B by $A^{1/2}$ in Theorems 2.1, 2.5 we get:

Corollay 2.6. *Let $A \geq 0$ and let $Z > 0$ with extremal eigenvalues a and b. Then, for all k,*

$$\frac{2\sqrt{ab}}{a+b}\,\lambda_k(AZ) \leq \mu_k(AZ) \leq \frac{a+b}{2\sqrt{ab}}\,\lambda_k(AZ).$$

Note that Corollary 2.6 contains Lemma 2.3, hence Lemma 2.2.

By replacing in Theorem 2.5 A and B by a rank one projection $h \otimes h$ we recapture the Kantorovich inequality of Lemma 2.2. This shows that Theorem 2.5 is sharp and, since they are equivalent, also Theorem 2.1 (see [8] for more details). Similarly to Theorem 2.5, the next theorem is also a sharp inequality extending Lemma 2.2.

Theorem 2.7. *Let A, B such that $AB \geq 0$ and let $Z > 0$ with extremal eigenvalues a and b. Then, for all symmetric norms,*

$$\|ZAB\| \leq \frac{a+b}{2\sqrt{ab}}\,\|BZA\|.$$

As in the special case of the operator norm (1), a basic rearrangement inequality for general symmetric norms claims that

$$\|AB\| \leq \|BA\| \tag{14}$$

whenever the product AB is normal. Thus, when $AB \geq 0$ Theorem 2.7 is a generalization of (14). Let us give a proof of (14). First for all symmetric norms and all partitionned matrices,

$$\left\|\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right\| \leq \left\|\begin{pmatrix} A & R \\ S & B \end{pmatrix}\right\|,$$

indeed, the left hand side is the mean of two unitary conguences of the right hand side,

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A & R \\ S & B \end{pmatrix} + \frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} A & R \\ S & B \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

By repetiton of this argument we see that symmetric norms of any matrix X are greater than those of its diagonal,

$$\|\mathrm{diag}(X)\| \leq \|X\|. \tag{15}$$

This inequality is quite important. Applying (15) to $X = BA$ with AB normal we deduce, by writing X in a triangular form, that

$$\|AB\| = \|\mathrm{diag}(BA)\| \leq \|BA\|.$$

Therefore (14) holds.

Proof of Theorem 2.7. Using Corollary 2.6 we have

$$\begin{aligned}\|ZAB\| = \|\mathrm{diag}(\mu_k(ZAB))\| &\leq \frac{a+b}{2\sqrt{ab}}\,\|\mathrm{diag}(\lambda_k(ZAB))\| \\ &= \frac{a+b}{2\sqrt{ab}}\,\|\mathrm{diag}(\lambda_k(BZA))\| \leq \frac{a+b}{2\sqrt{ab}}\,\|BZA\|\end{aligned}$$

where the last inequality follows from (15) applied to BZA in a triangular form. $\square$

Remarks. Starting from Lemma 2.2, we first proved Theorem 2.7 in [5] by using Ky Fan dominance principle (see the next section). As applications we then derived the above Lemmas 2.3 and 2.4. Theorem 2.1 had been proved later [8]. In some sens, the presentation given here, which starts from the earlier Theorem 1.5, is more natural. From Lemma 2.4 we also derived:
(Mond-Pečarić [22]) *Let $Z > 0$ with extremal eigenvalues a and b. Then, for every subspace $\mathcal{E}$,*

$$(Z_{\mathcal{E}})^{-1} \geq \frac{4ab}{(a+b)^2}\,(Z^{-1})_{\mathcal{E}}.$$

A similar compression inequality holds for others operator convex functions [5]. Mond-Pečarić's result is clearly an extension of the original Kantorovich inequality (6), (7).

Corollary 2.8. *Let A, $B > 0$ with $AB = BA$ and $pI \geq AB^{-1} \geq qI$ for some $p, q > 0$. Then, for all $Z \geq 0$ and all symmetric norms*

$$\|AZB\| \leq \frac{p+q}{2\sqrt{pq}}\,\|ZAB\|.$$

Proof. Write $AZB = AZA(A^{-1}B)$ and apply Theorem 2.7 with $A^{-1}B$ instead of Z. $\square$

2.1. Rearrangement inequalities for sequences

Corollary 2.8 can not be extended to normal operators Z, except in the case of the trace norm. This observation leaded to establish [9] the following reverse

inequality to the most basic rearrangement inequality (9). Recall that down arrows mean nonincreasing rearrangements.

Theorem 2.9. *Let* $\{a_i\}_{i=1}^n$ *and* $\{b_i\}_{i=1}^n$ *be* n*-tuples of positive numbers with*

$$p \geq \frac{a_i}{b_i} \geq q, \quad i = 1, \dots, n,$$

for some $p, q > 0$*. Then,*

$$\sum_{i=1}^n a_i^{\downarrow} b_i^{\downarrow} \leq \frac{p+q}{2\sqrt{pq}} \sum_{i=1}^n a_i b_i.$$

Proof. Introduce the diagonal matrices $A = \operatorname{diag}(a_i)$ and $B = \operatorname{diag}(b_i)$ and observe that, $\|\cdot\|_1$ standing for the trace norm,

$$\sum_{i=1}^n a_i b_i = \|AB\|_1$$

and

$$\sum_{i=1}^n a_i^{\downarrow} b_i^{\downarrow} = \|AVB\|_1$$

for some permutation matrix V. Hence we have to show that

$$\|AVB\|_1 \leq \frac{p+q}{2\sqrt{pq}} \|AB\|_1$$

To this end consider the spectral representation $V = \sum_i v_i h_i \otimes h_i$ where v_i are the eigenvalues and h_i the corresponding unit eigenvectors. We have

$$\begin{aligned}
\|AVB\|_1 &\leq \sum_{i=1}^n \|A \cdot v_i h_i \otimes h_i \cdot B\|_1 \\
&= \sum_{i=1}^n \|Ah_i\| \, \|Bh_i\| \\
&\leq \frac{p+q}{2\sqrt{pq}} \sum_{i=1}^n \langle Ah_i, Bh_i \rangle \\
&= \frac{p+q}{2\sqrt{pq}} \sum_{i=1}^n \langle h_i, ABh_i \rangle \\
&= \frac{p+q}{2\sqrt{pq}} \|AB\|_1
\end{aligned}$$

where we have used the triangle inequality for the trace norm and Lemma 2.10 below. $\square$

Lemma 2.10 *Let A, $B > 0$ with $AB = BA$ and $pI \geq AB^{-1} \geq qI$ for some $p, q > 0$. Then, for every vector h,*

$$\|Ah\| \, \|Bh\| \leq \frac{p+q}{2\sqrt{pq}} \langle Ah, Bh \rangle.$$

Proof. Write $h = B^{-1}f$ and apply Lemma 2.2; or apply Corollary 2.8 with $Z = h \otimes h$. $\square$

Remark. Lemma 2.10 extends Lemma 2.2 and is nothing less but of Cassel's Inequality:

Cassel's inequality. For nonnegative n-tuples $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$ and $\{w_i\}_{i=1}^n$ with

$$p \geq \frac{a_i}{b_i} \geq q, \quad i = 1, \ldots, n,$$

for some p, $q > 0$; it holds that

$$\left(\sum_{i=1}^n w_i a_i^2\right)^{1/2} \left(\sum_{i=1}^n w_i b_i^2\right)^{1/2} \leq \frac{p+q}{2\sqrt{pq}} \sum_{i=1}^n w_i a_i b_i.$$

Of course it is a reverse inequality to the Cauchy-Schwarz inequality. To obtain it from Lemma 2.9, one just takes $A = \mathrm{diag}(a_1, \ldots, a_n)$, $B = \mathrm{diag}(b_1, \ldots, b_n)$ and $h = (\sqrt{w_1}, \ldots, \sqrt{w_n})$. If one let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ then Cassel's inequality can be written

$$\|a\| \, \|b\| \leq \frac{p+q}{2\sqrt{pq}} \langle a, b \rangle$$

for a suitable inner product $\langle \cdot, \cdot \rangle$. It is then natural to search for conditions on a, b ensuring that the above inequality remains valid with Ua, Ub for all orthogonal matrices U. This motivates a remarkable extension of Cassel's inequality:

Dragomir's inequality. For real vectors a, b such that $\langle a - qb, pb - a \rangle \geq 0$ for some scalars p, q with $pq > 0$, inequality (1) holds.

Dragomir's inequality admits a version for complex vectors. For these inequalities see [10], [11], [12].

Remark. In [9] we also investigate reverse additive inequalities to (9). This setting is less clear. In general reverse additive type inequalities are more difficult than multiplicative ones. The story of Ozeki's inequality, a reverse additive inequality to Cauchy-Schwarz's inequality illustrates that [19].

3. Generalized Kantorovich inequalities

In [1] (see also [2, pp. 258, 285]) Araki showed a trace inequality which entails the following inequality for symmetric norms:

Theorem 3.1. *Let $A \geq 0$, $Z \geq 0$ and $p > 1$. Then, for every symmetric norm,*

$$\|(AZA)^p\| \leq \|A^p Z^p A^p\|.$$

For $0 < p < 1$, the above inequality is reversed.

If we take a rank one projection $A = h \otimes h$, $\|h\| = 1$, then Araki's inequality reduces to Jensen's inequality for $t \longrightarrow t^p$,

$$\langle h, Zh\rangle^p \leq \langle h, Z^p h\rangle. \tag{16}$$

This inequality admits a reverse inequality. Ky Fan [20] introduced the following constant, for $a, b > 0$ and integers p,

$$K(a,b,p) = \frac{a^p b - ab^p}{(p-1)(a-b)} \left(\frac{p-1}{p} \frac{a^p - b^p}{a^p b - ab^p} \right)^p.$$

Furuta extended it to all real numbers (see for instance [17], [18]) and showed the sharp reverse inequality of (16): *If $Z > 0$ have extremal eigenvalues a and b, then*

$$\langle h, Z^p h\rangle \leq K(a,b,p)\langle h, Zh\rangle^p \tag{17}$$

for $p > 1$ and $p < 0$.

In a recent paper [14], Fujii-Seo-Tominaga extended (17) to an operator norm inequality: *For $A \geq 0$, $Z > 0$ with extremal eigenvalues a and b, and $p > 1$,*

$$\|A^p Z^p A^p\| \leq K(a,b,p)\|(AZA)^p\|_\infty.$$

Inspired by this result, we showed in [7]:

Theorem 3.2. *Let $A \geq 0$ and let $Z > 0$ with extremal eigenvalues a and b. Then, for every $p > 1$, there exist unitaries U, V such that*

$$\frac{1}{K(a,b,p)} U(AZA)^p U^* \leq A^p Z^p A^p \leq K(a,b,p) V(AZA)^p V^*.$$

The Ky Fan constant $K(a,b,p)$ and its inverse are optimal.

For $p = 2$, Theorem 3.2 is a reformulation of Corollary 2.6.

Furuta introduced another constant depending on reals a, b and $p > 1$

$$C(a,b,p) = (p-1)\left(\frac{a^p - b^p}{p(a-b)}\right)^{p/(p-1)} + \frac{ab^p - ba^p}{a-b}$$

in order to obtain

$$\langle h, Z^p h\rangle - \langle h, Zh\rangle^p \leq C(a,b,p) \tag{18}$$

for unit vectors h and $Z \geq 0$ with extremal eigenvalues a and b (see [24, Theorem C]). Equivalently,

$$C(a,b,p) = \max\{\int_\Omega f^p \,\mathrm{d}\mu - \left(\int_\Omega f \,\mathrm{d}\right)^p\}$$

where the maximum runs over all measurable functions f, $a \geq f(t) \geq b$, on probabilized space (Ω, μ). Hence $a \geq a' \geq b' \geq b \Rightarrow C(a,b,p) \geq C(a',b',p)$.

Of course (18) generalizes the quadratic case (4) and $C(a,b,2) = (a-b)^2/4$. Simplified proofs are given in [14] by using the Mond-Pečarić method. It is also possible to prove it by reduction to the 2×2 matrix case, in a similar way of Lemma 2.2.

Furuta's constant allows us to extend the second inequality of Proposition 1.8 (in which $p = 2$):

Lemma 3.3. *Let $A \geq 0$ and let $Z \geq 0$ with extremal eigenvalues a and b. Then, for all $p > 1$,*

$$\mathrm{Tr}\, A^p Z^p A^p - \mathrm{Tr}\,(AZA)^p \leq C(a,b,p)\mathrm{Tr}\, A^{2p}.$$

This trace inequality can be extended to all symmetric norms:

Theorem 3.4. *Let $A \geq 0$ and let $Z \geq 0$ with extremal eigenvalues a and b. Then for all symmetric norms and all $p > 1$,*

$$\|A^p Z^p A^p\| - \|(AZA)^p\| \leq C(a,b,p)\|A^{2p}\|.$$

Note that letting A be a rank one projection either in the lemma or the theorem, we recapture inequality (18).

We will use the Ky Fan dominance principle: $\|A\| \leq \|B\|$ *for all symmetric norms iff* $\|A\|_{(k)} \leq \|B\|_{(k)}$ *for all Ky Fan k-norms.* By definition $\|A\|_{(k)}$ is the sum of the k largest singular values of A. For three different instructive proofs we refer to [2], [23] and [25, p. 56]. We also recall that

$$\|A\|_{(k)} = \max_E \|AE\|_1$$

where E runs over the set of rank k projections and $\|\cdot\|_1$ is the trace norm.

Proof of Lemma 3.3. Let $\{e_i\}$ be an orthonormal basis of eigenvectors for A and $\{a_i\}$ the corresponding aeigenvalues. Letting $\|\cdot\|_p$ denote Schatten p-norms and using the fact that the norm of the diagonal is less than the norm of the full matrix,

$$\begin{aligned}\mathrm{Tr}\, A^p Z^p A^p - \mathrm{Tr}\,(AZA)^p &= \mathrm{Tr}\, A^p Z^p A^p - \|AZA\|_p^p \\ &\leq \sum_i a_i^{2p}\langle e_i, Z^p e_i\rangle - \sum_i a_i^{2p}\langle e_i, Ze_i\rangle^p \\ &\leq C(a,b,p)\mathrm{Tr}\, A^{2p}\end{aligned}$$

where the second inequality follows from (18). □

Proof of Theorem 3.4. The main step consists in showing that the result holds for each Ky Fan k-norm,

$$\|A^p Z^p A^p\|_{(k)} - \|(AZA)^p\|_{(k)} \leq C(a,b,p)\|A^{2p}\|_{(k)}. \tag{19}$$

To this end, note that there exists a rank k projection E such that

$$\begin{aligned}\|A^pZ^pA^p\|_{(k)} &= \|EA^pZ^pA^pE\|_1\\ &= \|Z^{p/2}A^pEA^pZ^{p/2}\|_1\\ &= \|(A^pEA^p)^{1/2}Z^p(A^pEA^p)^{1/2}\|_1.\end{aligned}$$

We may then apply Lemma 3.3 to get

$$\|A^pZ^pA^p\|_{(k)} \le \|\{(A^pEA^p)^{1/2p}Z(A^pEA^p)^{1/2p}\}^p\|_1 + C(a,b,p)\|A^pEA^p\|_1.$$

Since $\|A^pE^pA^p\|_1 = \|EA^{2p}E\|_1 \le \|A^{2p}\|_{(k)}$ it suffices to show

$$\|\{(A^pEA^p)^{1/2p}Z(A^pEA^p)^{1/2p}\}^p\|_1 \le \|(AZA)^p\|_{(k)}$$

or equivalently

$$\|\{Z^{1/2}(A^pEA^p)^{1/p}Z^{1/2}\}^p\|_1 \le \|(Z^{1/2}A^2Z^{1/2})^p\|_{(k)}. \tag{20}$$

Let $X = Z^{1/2}(A^pEA^p)^{1/p}Z^{1/2}$ and $Y = Z^{1/2}A^2Z^{1/2}$. Since $t \longrightarrow t^{1/p}$ is operator monotone we infer $X \le Y$. Next we note that there exists a rank k projection F such that $FX = XF$ and $\|X^p\|_1 = \|FX^pF\|_1$. Hence we may apply the auxillary lemma below to obtain

$$\|X^p\|_1 \le \|FY^pF\|_1$$

which is the same as (20).

Having proved (19), let us show the general case. We first write (19) as

$$\|A^pZ^pA^p\|_{(k)} \le \|(AZA)^p\|_{(k)} + C(a,b,p)\|A^{2p}\|_{(k)} \tag{21}$$

and we introduce a unitary V such that $(AZA)^p$ and $VA^{2p}V^*$ form a monotone pair. Then (21) is equivalent to

$$\begin{aligned}\|A^pZ^pA^p\|_{(k)} &\le \|(AZA)^p\|_{(k)} + C(a,b,p)\|VA^{2p}V^*\|_{(k)}\\ &= \|(AZA)^p + C(a,b,p)VA^{2p}V^*\|_{(k)}\end{aligned}$$

by the simple fact that $\|X+Y\|_{(k)} = \|X\|_{(k)} + \|Y\|_{(k)}$ for all positive monotone pairs (X,Y). Therefore, Fan's dominance principle entails

$$\|A^pZ^pA^p\| \le \|(AZA)^p + C(a,b,p)VA^{2p}V^*\|$$

and the triangular inequality completes the proof. $\square$

Lemma 3.5. *Let $0 \le X \le Y$ and let F be a projection, $FX = XF$. Then, for all $p > 1$,*

$$\mathrm{Tr}\, FX^pF \le \mathrm{Tr}\, FY^pF.$$

Proof. Compute $\mathrm{Tr}\, FX^pF$ in a basis of eigenvectors for XF and apply (16). $\square$

Remark. In [14, 16] several results related to Theorem 3.2, 3.4 are given for the operator norm. For instance, in [16] the authors prove Theorem 3.4 ([16, Corollary 9]) and give results for $0 < p < 1$. A special case of resuls in [14] is an additive

version of Lemma 2.2. Under the same assumptions of this lemma, the authors show:

$$\|AZ\|_\infty - \rho(AZ) \le \frac{(a-b)^2}{4(a+b)}\|A\|_\infty.$$

Of course, letting A be a rank one projection we recapture a classical reverse inequality, companion to (4). Finally, let us mention a new book in which the reader may find many other reverse inequalities and references, T. Furuta, J. Mićić, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities,* Monograph in Inequalities 1, Element, Zagreb, 2005.

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